Since $m_{st} \propto I^2a$, and $m_{sc} \propto Ia$, then, whatever the size, there is a critical current below which $m_{st} < m_{sc}$. Using the structural material chosen by Bhattacharjie and Michael and a critical current density of 10^5 amp/cm^2, this current is roughly 10^7 amp. Since this is 100 times greater than the current required for shielding, we expect the structural weight to be about 1% of the superconducting material weight, independent of the size. The whole question of structure is, therefore, unimportant in the context of 10 Mev electrons.

Using the value $a/c_{st} = 0.6$ taken from Ref. 2 and using the same reference to estimate the true shielded volume for a coil of this type, one arrives at the conclusion that the weight of the shield lies almost entirely in the superconducting material; this weight is roughly given by

$$m_{sc} = 25V^{1/3}$$

The structural weight is a negligible fraction of the superconducting weight. Note the correct power dependence of m on V, when the optimization is done at constant current. These weights are lower than those quoted by Bhattacharjie and Michael by a factor of 3 when $V = 10^2$, and by a factor of 50 when $V = 10^5$. With weights as low as these, it is likely that considerations of the area of cryogenic surface required would exert a significant influence on the design.

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Comments on "Electron Fluctuations in an Equilibrium Turbulent Plasma"

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THE analysis according to which Demetriades[1] obtains expressions for the "proper" average electron density and the rms fluctuation of electron density for a turbulent plasma in thermochemical equilibrium includes an assumption that implies a particular probability distribution for the temperature. As will be shown, this distribution consists of two delta functions, each one being displaced from the average temperature by the rms temperature fluctuation. Thus a special case of the "marble cake" model[3, 4] is implied by Demetriades' assumption.

The probability distribution can be derived according to standard methods as follows. Let $P(T)$ be the probability distribution in T space; then the expected value $\langle f(T)\rangle$ of any function $f(T)$ is

$$\langle f(T)\rangle = \int_0^\infty f(T)\,P(T)\,dT \tag{1}$$

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Now consider the Fourier cosine transform of $P(T)$

$$P^*(\lambda) = \int_0^\infty P(T)\cos\lambda T\,dT \tag{2}$$

and let

$$\bar{T} = \langle T\rangle = \int_0^\infty TP(T)\,dT$$

Then Eq. (2) may be written as

$$\begin{aligned}P^*(\lambda) &= \int_0^\infty P(T)\cos\lambda(T - \bar{T} + \bar{T})\,dT\\ &= \int_0^\infty P(T)[\cos\lambda(T-\bar{T})\cos\lambda\bar{T} - \sin\lambda(T-\bar{T})\times \sin\lambda\bar{T}]dT\\ &= \int_0^\infty P(T)\left\{\cos\lambda\bar{T}\left[1 - \frac{\lambda^2(T-\bar{T})^2}{2!} + \frac{\lambda^4(T-\bar{T})^4}{4!} + \ldots\right] - \sin\lambda\bar{T}\left[\frac{\lambda(T-\bar{T})}{1!} - \frac{\lambda^3(T-\bar{T})^3}{3!} + \ldots\right]\right\}dT\end{aligned}$$

Therefore,

$$P^*(\lambda) = \cos\lambda\bar{T}[1 - (\lambda^2/2!)\langle(T-\bar{T})^2\rangle + \ldots] - \sin\lambda\bar{T}[-(\lambda^3/3!)\langle(T-\bar{T})^3\rangle + \ldots] \tag{3}$$

If one now invokes Demetriades'[1] assumptions, viz.,

$$\langle(T-\bar{T})^m\rangle = \begin{cases}0 & m = \text{odd integer}\\ [\langle(T-\bar{T})^2\rangle]^{m/2} & m = \text{even integer}\end{cases}$$

then Eq. (3) becomes

$$\begin{aligned}P^*(\lambda) &= \cos\lambda\bar{T}\left[1 - \frac{\lambda^2}{2!}\langle(T-\bar{T})^2\rangle + \frac{\lambda^4}{4!}\langle(T-\bar{T})^2\rangle^2 + \ldots\right]\\ &= \cos\lambda\bar{T}\cos\lambda[\langle(T-\bar{T})^2\rangle]^{1/2}\end{aligned} \tag{4}$$

Inverting, there results for $P(T)$

$$P(T) = \frac{2}{\pi}\int_0^\infty \cos\lambda\bar{T}\cos\lambda[\langle(T-\bar{T})^2\rangle]^{1/2}\cos\lambda T\,d\lambda$$

or

$$P(T) = \frac{1}{\pi}\int_0^\infty \cos\lambda T\,(\cos\lambda\{\bar{T} + [\langle(T-\bar{T})^2\rangle]^{1/2}\} + \cos\lambda\{\bar{T} - [\langle(T-\bar{T})^2\rangle]^{1/2}\})\,d\lambda \tag{5}$$

Therefore,[2]

$$P(T) = \tfrac{1}{2}[\delta(T - \{\bar{T} + [\langle(T-\bar{T})^2\rangle]^{1/2}\}) + \delta(T - \{\bar{T} - [\langle(T-\bar{T})^2\rangle]^{1/2}\})] \tag{6}$$

Thus, if physical significance is to be assigned to the results of (1), then only the temperatures $\bar{T} \pm [\langle(T-\bar{T})^2\rangle]^{1/2}$ are allowed to occur in the physical problem. This is physically unrealistic, and, accordingly, quantitative information based on the results of (1) should be used with caution. However, it does correspond to the special case of the marble-cake model in which equal weightings occur for "hot" and "cold" constituents.

If a distribution such as Eq. (6) is accepted, then all statistical properties of any function of T (i.e., expected values, rms deviations therefrom, etc.) may be immediately evaluated without recourse to further approximations. For example, applying Eq. (6) to electron density n_e as a function of temperature T, there results for the average electron density

$$\frac{\langle n_e(T)\rangle}{n_e(\langle T\rangle)} = \frac{\overline{n_e(T)}}{n_e(\bar{T})} = \frac{1}{2}\left(\frac{n_e\{\bar{T} + [\overline{(\Delta T)^2}]^{1/2}\} + n_e\{\bar{T} - [\overline{(\Delta T)^2}]^{1/2}\}}{n_e(\bar{T})}\right) \tag{7}$$

whereas the rms fluctuation is given by

$$\frac{\langle [n_e(T) - \langle n_e(T)\rangle]^2\rangle^{1/2}}{\langle n_e(T)\rangle} = \frac{[\overline{(\Delta n_e)^2}]^{1/2}}{\overline{n_e(T)}} = \frac{|n_e\{\bar{T} + [\overline{(\Delta T)^2}]^{1/2}\} - n_e\{\bar{T} - [\overline{(\Delta T)^2}]^{1/2}\}|}{n_e\{\bar{T} + [\overline{(\Delta T)^2}]^{1/2}\} + n_e\{\bar{T} - [\overline{(\Delta T)^2}]^{1/2}\}} \tag{8}$$

in which $[\overline{(\Delta T)^2}]^{1/2} = [\langle (T - \bar{T})^2\rangle]^{1/2}$.

Using the Saha equation and neglecting density fluctuation,[1]

$$\frac{\overline{n_e(T)}}{n_e(\bar{T})} = \frac{1}{2}\left\{(1+\epsilon)^{3/4}\exp\left[\frac{B}{2\bar{T}}\frac{\epsilon}{(1+\epsilon)}\right] + (1-\epsilon)^{3/4}\exp\left[-\frac{B}{2\bar{T}}\frac{\epsilon}{(1-\epsilon)}\right]\right\} \tag{9}$$

$$\frac{[\overline{(\Delta n_e)^2}]^{1/2}}{\overline{n_e(T)}} = \frac{\left|(1+\epsilon)^{3/4}\exp\left[\frac{B}{2\bar{T}}\frac{\epsilon}{(1+\epsilon)}\right] - (1-\epsilon)^{3/4}\exp\left[-\frac{B}{2\bar{T}}\frac{\epsilon}{(1-\epsilon)}\right]\right|}{(1+\epsilon)^{3/4}\exp\left[\frac{B}{2\bar{T}}\frac{\epsilon}{(1+\epsilon)}\right] + (1-\epsilon)^{3/4}\exp\left[-\frac{B}{2\bar{T}}\frac{\epsilon}{(1-\epsilon)}\right]} \tag{10}$$

in which $\epsilon = [\overline{(\Delta T)^2}]^{1/2}/\bar{T}$, and B is the ionization energy. When $\epsilon \ll 1$, Eqs. (9) and (10) take the respective forms

$$\frac{\overline{n_e(T)}}{n_e(\bar{T})} = \cosh\frac{B\epsilon}{2\bar{T}} + 0(\epsilon) \tag{9a}$$

$$\frac{\left[\overline{(\Delta n_e)^2}\right]^{1/2}}{\overline{n_e(T)}} = \tanh\frac{B\epsilon}{2\bar{T}} + 0(\epsilon^2) \tag{10a}$$

It is seen that, to lowest order, the preceding are identical to Eqs. (16) and (19) of Ref. 1.

In summary, the results of Ref. 1 concerning the values of the ratios $\overline{n_e(T)}/n_e(\bar{T})$ and $\left[\overline{(n_e)^2}\right]^{1/2}/\overline{n_e(T)}$ have been shown to be based on a physically unrealistic probability distribution for the temperature, the distribution consisting of two equally weighted delta functions; i.e., a special case of the marble-cake model. It then follows that the analysis of Ref. 1 is equivalent to simply assuming that, for any function $f(T)$, the expected value is the arithmetic average of $f\{\bar{T} + [\overline{(\Delta T)^2}]^{1/2}\}$ and $f\{\bar{T} - [\overline{(\Delta T)^2}]^{1/2}\}$, and that the rms fluctuation is one-half of the magnitude of the difference of $f\{T + [\overline{(\Delta T)^2}]^{1/2}\}$ and $f\{T - [\overline{(\Delta T)^2}]^{1/2}\}$.

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Reply by Author to F. Lane and S. L. Zeiberg

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THIS writer is indebted to Lane and Zeiberg for their observation on the probability distribution implied by the results of Ref. 1. In fact, appropriate reservations on the distribution implied by Eq. (12) of the latter reference were also voiced in the same paper, although the form of the distribution itself was not derived.

The applicability of the results of Ref. 1 to the inviscid-mixing or marble-cake model[2,3] had also occurred to the writer. However, whereas in its early versions the latter model indicated no upper limit in the normalized electron density fluctuations (see Fig. 9 of Ref. 3), the results of Ref. 1 showed a definite upper limit (of unity) of the normalized fluctuations. A tentative explanation of this discrepancy is that the marble-cake formulation is unable to distinguish between the pseudoaverage $n_e(\bar{T})$ and the numerically much larger proper average $\overline{n_e(T)}$. Until the latter formulation is completed to the point where it can be used to compute electron-density fluctuations in a more definitive sense, its identification with the results of Ref. 1 may be misleading.

Further calculations, such as those shown in Ref. 1, have been performed at this laboratory using a Gaussian, a parabolic, and a square probability distribution function. The results are shown in Fig. 1. These indicate that, for all reasonably well-behaved distributions, the fluctuation $[\overline{(\Delta n_e)^2}]^{1/2}/\overline{n_e(T)}$

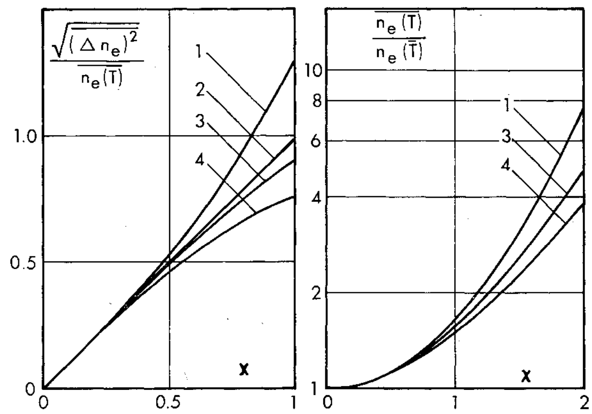

Fig. 1 Dependence of electron density fluctuations (left) and proper average density (right) on virtual temperature fluctuation for various probability distributions: 1) Gaussian, 2) parabolic, 3) square, 4) Ref. 1, Eq. (12).

is proportional to the virtual fluctuation χ for small values of the latter; for air, this corresponds to temperature fluctuations below about 3% for a temperature of 3000°K. This is understandable because for small χ the computation of the higher-order correlations, and, hence, the use of a distribution function, are unnecessary [see Eq. (15), Ref. 1]. However, in contrast to the results of Ref. 1, the previously mentioned distributions do not force the electron fluctuations to any limit as χ increases. On the other hand, all such distribution functions show that the difference between the pseudoaverage and the proper average electron density is even larger than that predicted by Ref. 1.

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